

Realizations of κ -Minkowski space, Drinfeld twists and related symmetry algebras

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Realizations of κ -Minkowski space linear in momenta are studied for time-, space- and light-like deformations. We construct and classify all such linear realizations and express them in terms of $\mathfrak{gl}(n)$ generators. There are three one-parameter families of linear realizations for time-like and space-like deformations, while for light-like deformations, there are only four linear realizations. The relation between deformed Heisenberg algebra, star product, coproduct of momenta and twist operator is presented. It is proved that for each linear realization there exists Drinfeld twist satisfying normalization and cocycle conditions. κ -deformed $\mathfrak{igl}(n)$ -Hopf algebras are presented for all cases. The κ -Poincaré-Weyl and κ -Poincaré-Hopf algebras are discussed. Left-right dual κ -Minkowski algebra is constructed from the transposed twists. The corresponding realizations are nonlinear. All Drinfeld twists related to κ -Minkowski space are obtained from our construction. Finally, some physical applications are discussed.

Keywords: noncommutative space, κ -Minkowski spacetime, Drinfeld twist, Hopf algebra.

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I. INTRODUCTION

One of the biggest problems in fundamental theoretical physics is a great difficulty to reconcile quantum mechanics and general theory of relativity in order to formulate consistent theory of quantum gravity. It is argued that at very high energies the gravitational effects can no longer be neglected and that the spacetime is no longer a smooth manifold, but rather a fuzzy, or better to say a noncommutative space [1]. Physical theories on such noncommutative manifolds require a new framework. This new framework is provided by noncommutative geometry [2]. In this framework, the search for generalized (quantum) symmetries that leave the physical action invariant leads to deformation of Poincaré symmetry, with κ -Poincaré symmetry being among the most extensively studied [3–6].

κ -deformed Poincaré symmetry is algebraically described by the κ -Poincaré-Hopf algebra and is an example of deformed relativistic symmetry that can possibly describe the physical reality at the Planck scale. κ is the deformation parameter usually interpreted as the Planck mass or some quantum gravity scale. It was shown that quantum field theory with κ -Poincaré symmetry emerges in a certain limit of quantum gravity coupled to matter fields after integrating out the gravitational/topological degrees of freedom [7]. This amounts to an effective theory in the form of a noncommutative field theory on the κ -deformed Minkowski space.

It is known [8] that the deformations of the symmetry group can be realized through the application of the Drinfeld twist on that symmetry group [9–12]. The main virtue of the twist formulation is that the deformed (twisted) symmetry algebra is the same as the original undeformed one and that there is only a change in the coalgebra structure which then leads to the same free field structure as the corresponding commutative field theory.

In [13] it was shown that the coproduct of $D = 2$ and $D = 4$ quantum κ -Poincaré algebra in the classical basis can not be obtained by the cochain twist depending only on the Poincaré generators (even if the coassociativity condition is relaxed). However the deformation used in [13] is the so called time-like type of deformation, and it is known [14] that for light-like deformation such a twist indeed exists [15–18].

In this work, we go other way round. Starting from κ -Minkowski space, we obtain its linear

realizations, then coproducts of momenta from realizations and finally, we present a method for obtaining corresponding twists from those coproducts. We show that, for linear realizations, those twists are Drinfeld twists, satisfying normalization and cocycle conditions. The method for obtaining Drinfeld twists corresponding to each linear realization is elaborated and it is shown how these twists generate new Hopf algebras. The resulting symmetry algebras are κ -deformed $\mathfrak{igl}(n)$ Hopf algebras. In special cases we obtain κ -Poincaré-Weyl-Hopf algebra and κ -Poincaré-Hopf algebra, but the former is obtained only for the case of light-like deformation.

The paper is organized as follows. In the second section, κ -Minkowski spacetime with deformation vector a_μ in various directions (timelike, spacelike and lightlike) is introduced. In section III, notion of linear realizations is introduced and all linear realizations in n dimensions for $n > 2$ are found. Those realizations are then expressed in terms of generators of $\mathfrak{gl}(n)$ algebra. In section IV, deformed Heisenberg algebra is presented, along with star product and coproducts of momenta. At the end of this section, the twist operator is introduced and the relation between star product, twist operator and coproduct of momenta is given. In section V it is shown that the twist operator from the previous section is a Drinfeld twist, satisfying normalization and cocycle conditions. It is shown that initial linear realizations follow from these twists, which confirms the consistency of our approach. At the end of the section V, \mathcal{R} -matrix is presented. In the section VI, κ -deformed $\mathfrak{igl}(n)$ Hopf algebra is presented, in general and for four special cases. In section VII, left-right dual κ -Minkowski algebra is constructed from the transposed twists. Alternatively, κ -Minkowski algebra is obtained from transposed twists with $a_\mu \rightarrow -a_\mu$. The corresponding realizations are non-linear. In section VIII, nonlinear realizations of κ -Minkowski space and related Drinfeld twists, known in the literature so far, are presented. Finally, in section IX, outlook and discussion are given.

II. κ -MINKOWSKI SPACE

κ -Minkowski space is usually defined by [3–5, 19]:

$$[\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa} \hat{x}_i, \quad [\hat{x}_i, \hat{x}_j] = 0. \quad (1)$$

Equations (1) can be rewritten in a covariant way [20]:

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu) \quad (2)$$

where $a_\mu \in \mathbb{M}^n$ (\mathbb{M}^n being undeformed n -dimensional Minkowski space) is a fixed deformation vector, which for the choice $a_0 = \kappa^{-1}$ and $a_i = 0$ corresponds to (1). Noncommutative coordinates \hat{x}_μ of κ -Minkowski space form a Lie algebra.

Note that Lie algebra (2) is independent of metric. However, we point out that our physical requirement is that in the limit $a_\mu \rightarrow 0$, we get ordinary Minkowski spacetime. Hence, it is natural to assume and treat a_μ as a vector in undeformed Minkowski space. There are two possibilities. One is to fix real parameters a_μ and the other is when a_μ are not fixed (transforming together with noncommutative coordinates \hat{x}_μ) [21]. In this paper we chose the first possibility.

Throughout the article, indices are raised and lowered by the Minkowski metric η , i.e. $a^\mu = \eta^{\mu\nu} a_\nu$ and $a_\mu = \eta_{\mu\nu} a^\nu$, where the convention with positive spatial eigenvalues of the metric is used, e.g. in $(3+1)$ dimensions $\eta = \text{diag}(-1, 1, 1, 1)$. Also, indices are contracted the same way, i.e. $a \cdot b = a_\mu b^\mu = \eta^{\mu\nu} a_\mu b_\nu$ and $a^2 = a \cdot a = a_\mu a^\mu = \eta^{\mu\nu} a_\mu a_\nu$ for any vectors a_μ and b_μ .

The deformation vector can be timelike ($a^2 < 0$), lightlike ($a^2 = 0$) and spacelike ($a^2 > 0$), so it can be written like:

$$a_\mu = \frac{1}{\kappa} u_\mu \quad (3)$$

where κ^{-1} is expansion parameter and $u^2 \in \{-1, 0, 1\}$, which corresponds to the previously mentioned three cases. Light-like deformation $a^2 = 0$ was first treated in context of null-plane quantum Poincaré algebra [22]. Depending on the sign of a^2 , κ -Minkowski Lie algebra is invariant under the following little groups:

- If a_μ is timelike ($a^2 < 0$), the little group is $SO(n-1)$
- If a_μ is lightlike ($a^2 = 0$), the little group is $E(n-2)$
- If a_μ is spacelike ($a^2 > 0$), the little group is $SO(n-2, 1)$

It is useful to introduce enveloping algebra $\hat{\mathcal{A}}$, generated by the elements \hat{x}_μ of κ -Minkowski algebra.

III. LINEAR REALIZATIONS

Commutative coordinates x_μ and momenta p_μ generate an undeformed Heisenberg algebra \mathcal{H} given by:

$$\begin{aligned} [x_\mu, x_\nu] &= 0, \\ [p_\mu, x_\nu] &= -i\eta_{\mu\nu}, \\ [p_\mu, p_\nu] &= 0 \end{aligned} \quad (4)$$

Analogously to $\hat{\mathcal{A}}$ in the previous section, commutative coordinates x_μ generate enveloping algebra \mathcal{A} , which is subalgebra of undeformed Heisenberg algebra, i.e. $\mathcal{A} \subset \mathcal{H}$. Momenta p_μ generate algebra \mathcal{T} , which is also subalgebra of undeformed Heisenberg algebra, i.e. $\mathcal{T} \subset \mathcal{H}$. Undeformed Heisenberg algebra is, symbolically, $\mathcal{H} = \mathcal{AT}$.

In general, realization of NC space is given by:

$$\hat{x}_\mu = x_\alpha \varphi^\alpha_\mu(p) \quad (5)$$

where $\varphi^\alpha_\mu(p)$ is a function of p_μ which should reduce to δ^α_μ in the limit when deformation goes to zero [23–25].

It is important to note that different realizations $\hat{x}_\mu = x_\alpha \varphi^\alpha_\mu(p) = x'_\alpha \varphi'^\alpha_\mu(p')$ are related by similarity transformations, where (x_μ, p_ν) and (x'_μ, p'_ν) satisfy undeformed Heisenberg algebra [26][27]. In this section, additional label for x_μ and p_μ is omitted for the sake of simplicity.

We are looking for linear realizations of κ -Minkowski space, that is the realizations where the function $\varphi^\alpha_\mu(p)$ is linear in p_μ . They can be written in the form

$$\hat{x}_\mu = x_\mu + l_\mu \quad (6)$$

where l_μ is linear in momentum p_μ . It is given by:

$$l_\mu = K_{\beta\mu}^\alpha x_\alpha p^\beta \quad (7)$$

where $K_{\beta\mu}^\alpha \in \mathbb{R}$. Inserting it in (2) gives that $K_{\mu\nu}^\alpha$ has to satisfy:

$$K_{\mu\nu}^\alpha - K_{\nu\mu}^\alpha = a_\mu \delta_\nu^\alpha - a_\nu \delta_\mu^\alpha \quad (8)$$

$$K_{\gamma\mu}^\alpha K_{\beta\nu}^\gamma - K_{\gamma\nu}^\alpha K_{\beta\mu}^\gamma = a_\mu K_{\beta\nu}^\alpha - a_\nu K_{\beta\mu}^\alpha \quad (9)$$

It also follows that l_μ satisfies the same commutation relations as \hat{x}_μ :

$$[l_\mu, l_\nu] = i(a_\mu l_\nu - a_\nu l_\mu) \quad (10)$$

A. Classification of linear realizations

Since we assume that equations (8) and (9) transform under Lorentz algebra, the most general covariant ansatz for $K_{\mu\nu}^\alpha$ in terms of deformation vector a_μ for arbitrary number of dimensions⁴ $n > 2$ is:

$$K_{\mu\nu}^\alpha = A_0 a_\mu a_\nu a^\alpha + A_1 \eta_{\mu\nu} a^\alpha + A_2 \delta_\mu^\alpha a_\nu + A_3 a_\mu \delta_\nu^\alpha \quad (11)$$

⁴ In 2 dimensions there are additional terms constructed with two dimensional Levi-Civita tensor $\epsilon_{\mu\nu}$. For example, there is a solution $K_{\mu\nu\lambda} = \frac{a_\mu a_\nu}{a^2} (c_1 a_\lambda + c_2 \epsilon_{\alpha\beta} a^\beta)$, where $c_1, c_2 \in \mathbb{R}$ are parameters and $a^2 \neq 0$.

leading to the following l_μ :

$$l_\mu = A_0 a_\mu (a \cdot x)(a \cdot p) + A_1 (a \cdot x) p_\mu + A_2 a_\mu (x \cdot p) + A_3 x_\mu (a \cdot p) \quad (12)$$

From equation (8) it follows that

$$A_3 = A_2 + 1. \quad (13)$$

Using (9) in combination with (13) yields the following equations:

$$A_1(A_0 a^2 + A_1 + 1) = 0 \quad (14)$$

$$A_3(A_0 a^2 + A_3 + 1) = 0 \quad (15)$$

$$A_1 A_3 a^2 = 0 \quad (16)$$

Those equations have four solutions:

1. $A_1 = 0, \quad A_2 = -1, \quad A_3 = 0, \quad a^2 A_0 = c$
2. $A_1 = 0, \quad A_2 = -c, \quad A_3 = 1 - c, \quad a^2 A_0 = c$
3. $A_1 = -1 - c, \quad A_2 = -1, \quad A_3 = 0, \quad a^2 A_0 = c$
4. $A_1 = -1, \quad A_2 = 0, \quad A_3 = 1, \quad a^2 = A_0 = 0$

where $c \in \mathbb{R}$ is a free parameter. We will denote these four types of realizations by $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and \mathcal{C}_4 respectively⁵. Explicitly for the tensor $K_{\mu\nu\alpha}$ we have

$$\begin{aligned} \mathcal{C}_1 : \quad K_{\mu\nu\alpha} &= \begin{cases} \frac{c}{a^2} a_\mu a_\nu a_\alpha - \eta_{\mu\alpha} a_\nu, & \text{if } a^2 \neq 0 \\ -\eta_{\mu\alpha} a_\nu, & \text{if } a^2 = 0, \end{cases} \\ \mathcal{C}_2 : \quad K_{\mu\nu\alpha} &= \begin{cases} \frac{c}{a^2} a_\mu a_\nu a_\alpha - c \eta_{\mu\alpha} a_\nu + (1 - c) \eta_{\nu\alpha} a_\mu, & \text{if } a^2 \neq 0 \\ \eta_{\nu\alpha} a_\mu, & \text{if } a^2 = 0, \end{cases} \\ \mathcal{C}_3 : \quad K_{\mu\nu\alpha} &= \begin{cases} \frac{c}{a^2} a_\mu a_\nu a_\alpha - (1 + c) \eta_{\mu\nu} a_\alpha - \eta_{\mu\alpha} a_\nu, & \text{if } a^2 \neq 0 \\ -\eta_{\mu\nu} a_\alpha - \eta_{\mu\alpha} a_\nu, & \text{if } a^2 = 0, \end{cases} \\ \mathcal{C}_4 : \quad K_{\mu\nu\alpha} &= -\eta_{\mu\nu} a_\alpha + \eta_{\nu\alpha} a_\mu, \quad \text{only for } a^2 = 0. \end{aligned} \quad (17)$$

⁵ Where \mathcal{C} stands for *covariant*.

Inserting (17) into (6) and (7) gives:

$$\begin{aligned}
\mathcal{C}_1: \quad \hat{x}_\mu &= \begin{cases} x_\mu + a_\mu \left[\frac{c}{a^2} (a \cdot x)(a \cdot p) - (x \cdot p) \right], & a^2 \neq 0 \\ x_\mu - a_\mu (x \cdot p), & a^2 = 0, \end{cases} \\
\mathcal{C}_2: \quad \hat{x}_\mu &= \begin{cases} x_\mu [1 + (1 - c)(a \cdot p)] + a_\mu \left[\frac{c}{a^2} (a \cdot x)(a \cdot p) - c(x \cdot p) \right], & a^2 \neq 0 \\ x_\mu [1 + (a \cdot p)], & a^2 = 0, \end{cases} \\
\mathcal{C}_3: \quad \hat{x}_\mu &= \begin{cases} x_\mu + a_\mu \left[\frac{c}{a^2} (a \cdot x)(a \cdot p) - (x \cdot p) \right] - (1 + c)(a \cdot x)p_\mu, & a^2 \neq 0 \\ x_\mu - a_\mu (x \cdot p) - (a \cdot x)p_\mu, & a^2 = 0, \end{cases} \\
\mathcal{C}_4: \quad \hat{x}_\mu &= x_\mu [1 + (a \cdot p)] - (a \cdot x)p_\mu, \quad \text{only for } a^2 = 0.
\end{aligned} \tag{18}$$

Linear realizations for κ -deformed Euclidean space were studied in [25]. However, in κ -Minkowski spacetime, we have found four new linear realizations corresponding to light-like deformations ($a^2 = 0$). Only one of them, \mathcal{C}_4 , corresponds to κ -Poincaré Hopf algebra [17]. \mathcal{C}_1 and \mathcal{C}_2 are equivalent for $c = 1$ while \mathcal{C}_1 and \mathcal{C}_3 are equivalent for $c = -1$. It is important to note that the first three solutions \mathcal{C}_1 , \mathcal{C}_2 and \mathcal{C}_3 are valid for all $a^2 \in \mathbb{R}$, and the fourth solution \mathcal{C}_4 is only valid in the case of light-like deformation.

The inverse matrices $\varphi_{\mu\nu}^{-1}$, such that $\varphi_\mu^\alpha \varphi_{\alpha\nu}^{-1} = \eta_{\mu\nu}$, for \mathcal{C}_1 , \mathcal{C}_2 , \mathcal{C}_3 and \mathcal{C}_4 are:

$$\begin{aligned}
\mathcal{C}_1: \quad \varphi_{\mu\nu}^{-1} &= \begin{cases} \eta_{\mu\nu} + \frac{1}{1 - (a \cdot p) + c} \left(p_\mu - c \frac{a_\mu}{a^2} \right) a_\nu, & a^2 \neq 0 \\ \eta_{\mu\nu} + \frac{p_\mu a_\nu}{1 - (a \cdot p)}, & a^2 = 0, \end{cases} \\
\mathcal{C}_2: \quad \varphi_{\mu\nu}^{-1} &= \begin{cases} \frac{\eta_{\mu\nu}}{1 + (1 - c)(a \cdot p)} - \frac{c}{[1 + (1 - c)(a \cdot p)]^2} \left(\frac{a \cdot p}{a^2} a_\mu + p_\mu \right) a_\nu, & a^2 \neq 0 \\ \frac{\eta_{\mu\nu}}{1 + (a \cdot p)}, & a^2 = 0, \end{cases} \\
\mathcal{C}_3: \quad \varphi_{\mu\nu}^{-1} &= \begin{cases} \eta_{\mu\nu} - \frac{1}{1 - a \cdot p(1 - c(a \cdot p))} \left[c(a \cdot p) \left(\frac{a_\mu}{a^2} + p_\mu \right) - p_\mu \right] a_\nu \\ + \frac{(1 + c) [(1 - a \cdot p)a_\mu + a^2 p_\mu] p_\nu}{1 - 2a \cdot p + (1 + c)((a \cdot p)^2 - a^2 p^2)}, & a^2 \neq 0 \\ \eta_{\mu\nu} + \frac{p_\mu a_\nu}{1 - a \cdot p} + \frac{[(1 - a \cdot p)a_\mu + a^2 p_\mu] p_\nu}{1 - 2a \cdot p + ((a \cdot p)^2 - a^2 p^2)}, & a^2 = 0, \end{cases} \\
\mathcal{C}_4: \quad \varphi_{\mu\nu}^{-1} &= \frac{\eta_{\mu\nu} + a_\mu p_\nu}{1 + (a \cdot p)}.
\end{aligned} \tag{19}$$

Special cases of \mathcal{C}_1 , \mathcal{C}_2 and \mathcal{C}_3 , when $c = 0$, we denote as \mathcal{S}_1 , \mathcal{S}_2 and \mathcal{S}_3 respectively:

$$\begin{aligned}
\mathcal{S}_1: \quad \hat{x}_\mu &= x_\mu - a_\mu (x \cdot p), \\
\mathcal{S}_2: \quad \hat{x}_\mu &= x_\mu [1 + (a \cdot p)], \\
\mathcal{S}_3: \quad \hat{x}_\mu &= x_\mu - a_\mu (x \cdot p) - (a \cdot x)p_\mu,
\end{aligned} \tag{20}$$

where $a^2 \in \mathbb{R}$.

B. Symmetry algebra $\mathfrak{igl}(n)$

For fixed solution $K_{\mu\nu\lambda}$ we define undeformed $\mathfrak{igl}(n)$ algebra generated by p_μ and $L_{\mu\nu}$:

$$\begin{aligned} [L_{\mu\nu}, L_{\lambda\rho}] &= \eta_{\nu\lambda} L_{\mu\rho} - \eta_{\mu\rho} L_{\lambda\nu} \\ [L_{\mu\nu}, p_\lambda] &= -p_\nu \eta_{\mu\lambda} \\ [p_\mu, p_\nu] &= 0 \end{aligned} \tag{21}$$

In addition to commutation relations (21),

$$[L_{\mu\nu}, x_\lambda] = x_\mu \eta_{\nu\lambda} \tag{22}$$

also holds.

Linear realizations can be written in terms of $L_{\mu\nu}$:

$$\hat{x}_\mu = x_\mu - i K_{\beta\mu\alpha} L^{\alpha\beta} \tag{23}$$

Particularly, for (18):

$$\begin{aligned} \mathcal{C}_1 : \quad \hat{x}_\mu &= \begin{cases} x_\mu - i a_\mu \left(\frac{c}{a^2} a_\alpha a_\beta L^{\alpha\beta} - L_\alpha^\alpha \right), & a^2 \neq 0 \\ x_\mu + i a_\mu L_\alpha^\alpha, & a^2 = 0, \end{cases} \\ \mathcal{C}_2 : \quad \hat{x}_\mu &= \begin{cases} x_\mu - i \frac{c}{a^2} a_\mu a_\alpha a_\beta L^{\alpha\beta} + i c L_\alpha^\alpha a_\mu - i(1-c) a^\alpha L_{\mu\alpha}, & a^2 \neq 0 \\ x_\mu - i a^\alpha L_{\mu\alpha}, & a^2 = 0, \end{cases} \\ \mathcal{C}_3 : \quad \hat{x}_\mu &= \begin{cases} x_\mu - i \frac{c}{a^2} a_\mu a_\alpha a_\beta L^{\alpha\beta} + i(1+c) a^\alpha L_{\alpha\mu} + i L_\alpha^\alpha a_\mu, & a^2 \neq 0 \\ x_\mu + i a^\alpha L_{\alpha\mu} + i L_\alpha^\alpha a_\mu, & a^2 = 0, \end{cases} \\ \mathcal{C}_4 : \quad \hat{x}_\mu &= x_\mu + i a^\alpha (L_{\alpha\mu} - L_{\mu\alpha}) = x_\mu + i a^\alpha M_{\alpha\mu}, \quad \text{only for } a^2 = 0 \end{aligned} \tag{24}$$

where $M_{\mu\nu} = L_{\mu\nu} - L_{\nu\mu}$ generate Lorentz algebra. Note that \mathcal{C}_4 is the only solution that can be written in terms of Lorentz generators.

Commutation relations between generators of $\mathfrak{igl}(n)$ algebra with \hat{x}_μ are

$$[p_\mu, \hat{x}_\nu] = -i \varphi_{\mu\nu} = -i(\eta_{\mu\nu} + K_{\alpha\nu\mu} p^\alpha) \tag{25}$$

$$[L_{\mu\nu}, \hat{x}_\lambda] = \hat{x}_\mu \eta_{\nu\lambda} + i(K_{\beta\lambda\alpha} \eta_{\nu\lambda} - K_{\beta\lambda\nu} \eta_{\alpha\mu} - K_{\mu\lambda\alpha} \eta_{\beta\nu}) L^{\alpha\beta} \tag{26}$$

Algebra generated by $L_{\mu\nu}$, p_μ and \hat{x}_μ , satisfies all the Jacobi relations. Only for solution \mathcal{C}_4 this is also true for algebra generated by $M_{\mu\nu}$, p_μ and \hat{x}_μ .

At the end of this section let us introduce anti-involution operator \dagger by $\lambda^\dagger = \bar{\lambda}$, for $\lambda \in \mathbb{C}$ and bar denoting the ordinary complex conjugation $(\hat{x}_\mu)^\dagger = \hat{x}_\mu$, $(x_\mu)^\dagger = x_\mu$, $(p_\mu)^\dagger = p_\mu$ and $(M_{\mu\nu})^\dagger = -M_{\mu\nu}$. Since $(a_\mu)^\dagger = a_\mu$ the relations (2), (4), (25) and (27) remain unchanged (i.e. they are invariant) under the action of \dagger . Note that realizations \mathcal{C}_1 , \mathcal{C}_2 and \mathcal{C}_3 are generally not hermitian. In order to get the hermitian realizations, one has to make following substitutions: $\hat{x}_\mu \rightarrow \frac{1}{2}(\hat{x}_\mu + \hat{x}_\mu^\dagger)$, $l_\mu \rightarrow \frac{1}{2}(l_\mu + l_\mu^\dagger)$, $L_{\mu\nu} \rightarrow \frac{1}{2}(L_{\mu\nu} - (L_{\mu\nu})^\dagger)$ throughout the whole paper [28].

IV. DEFORMED HEISENBERG ALGEBRA, STAR PRODUCT AND TWIST OPERATOR

Non-commutative κ -Minkowski coordinates \hat{x}_μ and momenta p_μ generate a deformed Heisenberg algebra $\hat{\mathcal{H}}$ given by [29]:

$$\begin{aligned} [\hat{x}_\mu, \hat{x}_\nu] &= i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu), \\ [p_\mu, \hat{x}_\nu] &= -i\varphi_{\mu\nu}(p) = -i(\eta_{\mu\nu} + K_{\alpha\nu\mu} p^\alpha), \\ [p_\mu, p_\nu] &= 0 \end{aligned} \tag{27}$$

From previous section, it follows that $\hat{\mathcal{H}}$ is isomorphic to \mathcal{H} . Algebra $\hat{\mathcal{A}}$ is a subalgebra of $\hat{\mathcal{H}}$, i.e. $\hat{\mathcal{A}} \subset \hat{\mathcal{H}}$. Deformed Heisenberg algebra is, symbolically, $\hat{\mathcal{H}} = \hat{\mathcal{A}}\mathcal{T}$.

A. Actions \blacktriangleright and \triangleright

Action \blacktriangleright is a map $\blacktriangleright: \hat{\mathcal{H}} \otimes \hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}}$ satisfying the following properties:

$$\hat{f} \blacktriangleright \hat{g} = \hat{f}\hat{g}, \quad \forall \hat{f}, \hat{g} \in \hat{\mathcal{A}} \tag{28}$$

$$p_\mu \blacktriangleright \hat{f} = [p_\mu, \hat{f}] \blacktriangleright 1, \quad \forall \hat{f} \in \hat{\mathcal{A}} \tag{29}$$

$$p_\mu \blacktriangleright 1 = 0 \tag{30}$$

It follows that

$$\begin{aligned} \hat{\mathcal{H}} \blacktriangleright 1 &= \hat{\mathcal{A}}, \\ \hat{\mathcal{A}} \blacktriangleright 1 &= \hat{\mathcal{A}}. \end{aligned} \tag{31}$$

In a complete analogy, the action \triangleright is a map $\triangleright : \mathcal{H} \otimes \mathcal{A} \rightarrow \mathcal{A}$ satisfying the following properties:

$$f \triangleright g = fg, \quad \forall f, g \in \mathcal{A} \quad (32)$$

$$p_\mu \triangleright f = [p_\mu, f] \triangleright 1, \quad \forall f \in \mathcal{A} \quad (33)$$

$$p_\mu \triangleright 1 = 0 \quad (34)$$

Also, it follows that

$$\begin{aligned} \mathcal{H} \triangleright 1 &= \mathcal{A}, \\ \mathcal{A} \triangleright 1 &= \mathcal{A}. \end{aligned} \quad (35)$$

\blacktriangleright and \triangleright are actions, so they satisfy:

$$(\hat{f}\hat{g}) \blacktriangleright \hat{h} = \hat{f} \blacktriangleright (\hat{g} \blacktriangleright \hat{h}) \quad (36)$$

$$(fg) \triangleright h = f \triangleright (g \triangleright h) \quad (37)$$

B. Star product

For κ -Minkowski space, there exists an isomorphism (as vector spaces) between $\hat{\mathcal{A}}$ and \mathcal{A} , defined by:

$$\begin{aligned} \hat{f} \triangleright 1 &= f \\ f \blacktriangleright 1 &= \hat{f}, \end{aligned} \quad (38)$$

where $\hat{f} \in \hat{\mathcal{A}}$, and also, using realization for \hat{x}_μ (5), $\hat{f} \in \mathcal{H}$. Similarly, $f \in \mathcal{A}$, and also inverting realization for \hat{x}_μ , $f \in \hat{\mathcal{H}}$.

Using this identification, star product $\star : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is defined by:

$$f \star g \equiv (\hat{f}\hat{g}) \triangleright 1 = \hat{f} \triangleright g \quad (39)$$

For κ -Minkowski space, the star product is associative:

$$(f \star g) \star h = f \star (g \star h). \quad (40)$$

Star product defines algebra \mathcal{A}_\star , which is defined like \mathcal{A} , but with non-commutative star product instead of ordinary multiplication. Algebras \mathcal{A}_\star and $\hat{\mathcal{A}}$ are isomorphic as algebras, not only as vector

spaces. It follows that:

$$(f \star g) \blacktriangleright 1 = \hat{f} \hat{g} \quad (41)$$

$$\hat{\mathcal{H}} \triangleright 1 = \mathcal{A}_\star, \quad (42)$$

$$\hat{\mathcal{A}} \triangleright 1 = \mathcal{A}_\star,$$

$$\mathcal{H} \blacktriangleright 1 = \hat{\mathcal{A}}, \quad (43)$$

$$\mathcal{A} \blacktriangleright 1 = \hat{\mathcal{A}}.$$

It can be shown that:

$$e^{ik \cdot \hat{x}} \triangleright 1 = e^{iK(k) \cdot x} \quad (44)$$

where $K_\mu(k)$ is invertible function $K_\mu : \mathbb{M}^n \rightarrow \mathbb{M}^n$, which is calculated in Appendix A for linear realizations of \hat{x}_μ . The inverse relation is

$$e^{ik \cdot x} \blacktriangleright 1 = e^{iK^{-1}(k) \cdot \hat{x}} \quad (45)$$

It can also be shown that there is a function $P_\mu(k_1, k_2)$, such that

$$e^{ik_1 \cdot \hat{x}} \triangleright e^{ik_2 \cdot x} = e^{iP(k_1, k_2) \cdot x} \quad (46)$$

The star product between such exponentials is then given by:

$$e^{ik_1 \cdot x} \star e^{ik_2 \cdot x} = e^{iK^{-1}(k_1) \cdot \hat{x}} \triangleright e^{ik_2 \cdot x} = e^{iP(K^{-1}(k_1), k_2) \cdot x} \equiv e^{i\mathcal{D}(k_1, k_2) \cdot x} \quad (47)$$

Note that $K_\mu(k) = P_\mu(k, 0)$ and $\mathcal{D}_\mu(k_1, k_2) = P_\mu(K^{-1}(k_1), k_2)$. $\mathcal{D}_\mu(k_1, k_2)$ describes deformed addition of momenta $(k_1)_\mu \oplus (k_2)_\mu = \mathcal{D}_\mu(k_1, k_2)$ (for more details see [30]). Calculation of $P_\mu(k_1, k_2)$, $\mathcal{D}_\mu(k_1, k_2)$ and $K_\mu(k)$ for linear realizations (described in previous section) is given in Appendix A.

For elements $f, g \in \mathcal{A}$ which can be Fourier transformed

$$f = \int d^n k \tilde{f}(k) e^{ik \cdot x} \quad (48)$$

$$g = \int d^n k \tilde{g}(k) e^{ik \cdot x} \quad (49)$$

we find corresponding elements $\hat{f}, \hat{g} \in \hat{\mathcal{A}}$

$$\hat{f} = f \blacktriangleright 1 = \int d^n k \tilde{f}(k) e^{iK^{-1}(k) \cdot \hat{x}} \quad (50)$$

$$\hat{g} = g \blacktriangleright 1 = \int d^n k \tilde{g}(k) e^{iK^{-1}(k) \cdot \hat{x}} \quad (51)$$

Then the star product $f \star g$ can be written in the following way:

$$f \star g = \hat{f} \hat{g} \triangleright 1 = \int d^n k_1 d^n k_2 \tilde{f}(k_1) \tilde{g}(k_2) e^{i\mathcal{D}(k_1, k_2) \cdot x} \quad (52)$$

C. Coproduct of momenta

The undeformed coproduct $\Delta_0 : \mathcal{T} \rightarrow \mathcal{T} \otimes \mathcal{T}$ for momentum p_μ is:

$$\Delta_0 p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu \quad (53)$$

Deformed coproduct for momenta $\Delta : \mathcal{T} \rightarrow \mathcal{T} \otimes \mathcal{T}$ is [31–33]:

$$\Delta p_\mu = \mathcal{D}_\mu(p \otimes 1, 1 \otimes p) \quad (54)$$

Using results from Appendix A, we have:

$$\Delta p_\mu = p_\mu \otimes 1 + \Lambda_{\alpha\mu}^{-1} \otimes p^\alpha \quad (55)$$

where

$$\Lambda_{\mu\nu} = (e^{\mathcal{K}})_{\mu\nu} \quad (56)$$

$$\mathcal{K}_{\mu\nu} = -K_{\mu\alpha\nu}(K^{-1})^\alpha(p) \quad (57)$$

and

$$\Delta \Lambda_{\mu\nu} = \Lambda_{\mu\alpha} \otimes \Lambda^\alpha{}_\nu \quad (58)$$

$$\Delta(\Lambda^{-1})_{\mu\nu} = (\Lambda^{-1})_{\alpha\nu} \otimes (\Lambda^{-1})_\mu{}^\alpha \quad (59)$$

We also have:

$$p_\mu \hat{f} = (p_\mu \blacktriangleright \hat{f}) + (\Lambda_{\alpha\mu}^{-1} \blacktriangleright \hat{f}) p^\alpha \quad (60)$$

$$\Lambda_{\mu\nu} \hat{f} = (\Lambda_{\mu\alpha} \blacktriangleright \hat{f}) \Lambda^\alpha{}_\nu \quad (61)$$

$$(\Lambda^{-1})_{\mu\nu} \hat{f} = ((\Lambda^{-1})_{\alpha\nu} \blacktriangleright \hat{f}) (\Lambda^{-1})_\mu{}^\alpha \quad (62)$$

For example if $\hat{f} = \hat{x}_\lambda$ we have

$$[\Lambda_{\mu\nu}, \hat{x}_\lambda] = i K_{\mu\lambda}{}^\alpha \Lambda_{\alpha\nu} \quad (63)$$

$$[\Lambda_{\mu\nu}^{-1}, \hat{x}_\lambda] = -i \Lambda_{\mu\alpha}^{-1} K^\alpha{}_{\lambda\nu} \quad (64)$$

In order to specify Δp_μ , we have to express $K_\mu^{-1}(p) \equiv p_\mu^W$ in terms of momenta p_μ (see Appendix A). Momentum p_μ acts on $e^{ik \cdot x}$ and $e^{ik \cdot \hat{x}}$ with \triangleright and \blacktriangleright respectively in the following way:

$$p_\mu \triangleright e^{ik \cdot x} = k_\mu e^{ik \cdot x}, \quad p_\mu \blacktriangleright e^{ik \cdot \hat{x}} = K_\mu(k) e^{ik \cdot \hat{x}} \quad (65)$$

and momentum p_μ^W acts as:

$$p_\mu^W \blacktriangleright e^{ik \cdot \hat{x}} = k_\mu e^{ik \cdot \hat{x}}, \quad p_\mu^W \triangleright e^{ik \cdot x} = K_\mu^{-1}(k) e^{ik \cdot x} \quad (66)$$

It is useful to introduce the shift operator Z , with properties:

$$[Z, \hat{x}_\mu] = i a_\mu Z \quad (67)$$

$$Z = e^{-a \cdot p^W} \quad (68)$$

Explicitly, for \mathcal{C}_1 , \mathcal{C}_2 , \mathcal{C}_3 and \mathcal{C}_4 , coproducts of momenta are:

- Case \mathcal{C}_1 :

$$\Delta p_\mu = p_\mu \otimes 1 + Z \otimes p_\mu + \frac{a_\mu}{a^2} (Z^{1-c} - Z) \otimes a \cdot p \quad (69)$$

$$\Lambda_{\mu\nu}^{-1} = \left[\eta_{\mu\nu} + \frac{a_\mu a_\nu}{a^2} (Z^{-c} - 1) \right] Z \quad (70)$$

$$\Lambda_{\mu\nu} = \left[\eta_{\mu\nu} + \frac{a_\mu a_\nu}{a^2} (Z^c - 1) \right] Z^{-1} \quad (71)$$

$$p_\mu^W = \left[p_\mu - \frac{a_\mu}{a^2} (Z - 1 + a \cdot p) \right] \frac{\ln Z}{Z - 1} \quad (72)$$

$$Z = [1 - (1 - c)a \cdot p]^{\frac{1}{1-c}} \quad (73)$$

- Case \mathcal{C}_2 :

$$\Delta p_\mu = p_\mu \otimes 1 + \left(Z^c - \frac{c}{1+c} \right) \otimes p_\mu + \left(c \frac{a_\mu}{a^2} + (c-1) \frac{p_\mu^W}{\ln Z} \right) \frac{Z^{-1} - Z^c}{1+c} \otimes a \cdot p \quad (74)$$

$$\Lambda_{\mu\nu}^{-1} = \eta_{\mu\nu} \left(Z^c - \frac{c}{1+c} \right) + a_\mu \left(c \frac{a_\nu}{a^2} + (c-1) \frac{p_\nu^W}{\ln Z} \right) \frac{Z^{-1} - Z^c}{1+c} \quad (75)$$

$$\Lambda_{\mu\nu} = \eta_{\mu\nu} \left(Z^{-c} - \frac{c}{1+c} \right) + a_\mu \left(c \frac{a_\nu}{a^2} + (c-1) \frac{p_\nu^W}{\ln Z} \right) \frac{Z - Z^{-c}}{1+c} \quad (76)$$

$$p_\mu^W = \left[p_\mu - \frac{a_\mu}{a^2} (1 - Z^{-1} + a \cdot p) \right] \frac{\ln Z}{1 - Z^{-1}} \quad (77)$$

$$Z = [1 - (c-1)a \cdot p]^{\frac{1}{c-1}} \quad (78)$$

- Case \mathcal{C}_3 :

$$\Delta p_\mu = p_\mu \otimes 1 + Z \otimes p_\mu + a_\mu \left((1+c) \frac{p_\mu^W}{\ln Z} - c \frac{a_\mu}{a^2} \right) (Z-1) Z \otimes p^\alpha \quad (79)$$

$$\Lambda_{\mu\nu}^{-1} = \left[\eta_{\mu\nu} + \left((1+c) \frac{p_\mu^W}{\ln Z} - c \frac{a_\mu}{a^2} \right) a_\nu (Z-1) \right] Z \quad (80)$$

$$\Lambda_{\mu\nu} = \left[\eta_{\mu\nu} + \left((1+c) \frac{p_\mu^W}{\ln Z} - c \frac{a_\mu}{a^2} \right) a_\nu (Z^{-1}-1) \right] Z^{-1} \quad (81)$$

$$p_\mu^W = \left[p_\mu - \frac{a_\mu}{a^2} (Z-1 + a \cdot p) \right] \frac{\ln Z}{Z-1} \quad (82)$$

$$Z = [c + (1-c)((1-a \cdot p)^2 - a^2 p^2)]^{\frac{1}{2(1-c)}} \quad (83)$$

- Case \mathcal{C}_4 :

$$\Delta p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu + p_\mu \otimes a \cdot p - a_\mu p_\alpha Z \otimes p^\alpha - \frac{a_\mu}{2} p^2 Z \otimes a \cdot p \quad (84)$$

$$\Lambda_{\mu\nu}^{-1} = \eta_{\mu\nu} + a_\mu p_\nu - \left(p_\mu + \frac{a_\mu}{2} p^2\right) a_\nu Z \quad (85)$$

$$\Lambda_{\mu\nu} = \eta_{\mu\nu} + p_\mu a_\nu - a_\mu \left(p_\nu + \frac{a_\nu}{2} p^2\right) Z \quad (86)$$

$$p_\mu^W = \left(p_\mu + \frac{a_\mu}{2} p^2\right) \frac{\ln Z}{1 - Z^{-1}} \quad (87)$$

$$Z = \frac{1}{1 + a \cdot p} \quad (88)$$

D. Relation between star product, twist operator and coproduct

The star product is related to the twist operator \mathcal{F}^{-1} in the following way:

$$f \star g = m \left[\mathcal{F}^{-1}(\triangleright \otimes \triangleright)(f \otimes g) \right] \quad (89)$$

where $f, g \in \mathcal{A}$. Furthermore,

$$\hat{f} = m \left[\mathcal{F}^{-1}(\triangleright \otimes 1)(f \otimes 1) \right], \quad f \in \mathcal{A} \quad (90)$$

where $\hat{f} \in \hat{\mathcal{A}}$ is expressed in terms of $x, p \in \mathcal{H}$.

Using the above expression for star product eq. (52) and (89), the twist operator can be written as [30, 34, 35]

$$\mathcal{F}^{-1} =: \exp \left[i(tx_\alpha \otimes 1 + (1-t) \otimes x_\alpha)(\Delta - \Delta_0)p^\alpha \right] : \quad (91)$$

where $t \in \mathbb{R}$ and is generally defined up to the right ideal $\mathcal{I}_0 \subset \mathcal{H} \otimes \mathcal{H}$ defined by

$$m(\mathcal{I}_0(\triangleright \otimes \triangleright)(\mathcal{A} \otimes \mathcal{A})) = 0 \quad (92)$$

V. DRINFELD TWISTS

Starting with expression (91) for twist operator, we derive Drinfeld twists [8–10] in Appendix B.

$$\mathcal{F} = \exp \left(\mathcal{K}_{\beta\alpha} \otimes L^{\alpha\beta} \right) = \exp \left(-ip_\alpha^W \otimes l^\alpha \right) \quad (93)$$

where p_μ^W is given in the subsection IV C after equation (64) and in Appendix A, and $l_\mu = -iK_{\beta\mu\alpha}L^{\alpha\beta}$, where $K_{\beta\mu\alpha}$ satisfies (8) and (9), with solutions eq. (17), l_μ generate κ -Minkowski algebra

$$[l_\mu, l_\nu] = i(a_\mu l_\nu - a_\nu l_\mu) \quad (94)$$

and $L_{\mu\nu}$ generate $\mathfrak{gl}(n)$ algebra, see equation (21).

The classical r-matrix r_{cl} , related to twist (93) is:

$$r_{cl} = p_\alpha \wedge l^\alpha = p_\alpha \otimes l^\alpha - l^\alpha \otimes p_\alpha \quad (95)$$

For $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and \mathcal{C}_4 , twists are:

$$\mathcal{F}_{\mathcal{C}_1} = \exp \left\{ - \left(\eta_{\alpha\beta} - c \frac{a_\alpha a_\beta}{a^2} \right) \ln Z \otimes L^{\alpha\beta} \right\} = \exp \left\{ - \ln Z \otimes \left(D - c \frac{a_\alpha a_\beta}{a^2} L^{\alpha\beta} \right) \right\}, \quad (96)$$

$$\mathcal{F}_{\mathcal{C}_2} = \exp \left\{ - \left[c \left(\eta_{\alpha\beta} - \frac{a_\alpha a_\beta}{a^2} \right) \ln Z + (1-c) a_\beta p_\alpha^W \right] \otimes L^{\alpha\beta} \right\}, \quad (97)$$

$$\mathcal{F}_{\mathcal{C}_3} = \exp \left\{ - \left[\left(\eta_{\alpha\beta} - c \frac{a_\alpha a_\beta}{a^2} \right) \ln Z - (1+c) a_\alpha p_\beta^W \right] \otimes L^{\alpha\beta} \right\}, \quad (98)$$

$$\mathcal{F}_{\mathcal{C}_4} = \exp \left\{ (a_\alpha p_\beta^W - a_\beta p_\alpha^W) \otimes L^{\alpha\beta} \right\} = \exp \left\{ a_\alpha p_\beta^W \otimes M^{\alpha\beta} \right\}, \quad a^2 = 0 \quad (99)$$

Note that only for the case \mathcal{C}_4 ($a^2 = 0$), the corresponding twist operator can be expressed in terms of Poincaré generators only [17, 36].

Starting from the twist operator, the realization can be obtained using

$$\hat{x}_\mu = m [\mathcal{F}^{-1}(\triangleright \otimes 1)(x_\mu \otimes 1)] = x_\mu - i K_{\beta\mu\alpha} L^{\alpha\beta} \quad (100)$$

Using twists (96), (97), (98) and (99) yields realizations $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and \mathcal{C}_4 respectively, which satisfy κ -Minkowski algebra.

A. Undeformed $\mathfrak{igl}(n)$ Hopf algebra

Coproducts $\Delta_0 : \mathfrak{igl}(n) \rightarrow \mathfrak{igl}(n) \otimes \mathfrak{igl}(n)$ in undeformed $\mathfrak{igl}(n)$ Hopf algebra are:

$$\Delta_0 p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu \quad (101)$$

$$\Delta_0 L_{\mu\nu} = L_{\mu\nu} \otimes 1 + 1 \otimes L_{\mu\nu}, \quad (102)$$

counit $\epsilon : \mathfrak{igl}(n) \rightarrow \mathbb{C}$ is

$$\epsilon(p_\mu) = \epsilon(p_\mu^W) = \epsilon(L_{\mu\nu}) = 0, \quad \epsilon(1) = 1 \quad (103)$$

and antipode $S_0 : \mathfrak{igl}(n) \rightarrow \mathfrak{igl}(n)$ is

$$S_0(p_\mu) = -p_\mu, \quad S_0(L_{\mu\nu}) = -L_{\mu\nu}. \quad (104)$$

B. Normalization condition

Now we show that these twists satisfy normalization condition and cocycle condition, i.e. that they are Drinfeld twists.

The normalization condition

$$m(\epsilon \otimes 1)\mathcal{F} = 1 = m(1 \otimes \epsilon)\mathcal{F} \quad (105)$$

follows trivially since the twist is of the form $\mathcal{F} = e^f$, where $f = -ip_\alpha^W \otimes l^\alpha$, therefore,

$$(\epsilon \otimes 1)f = (1 \otimes \epsilon)f = 0, \quad (106)$$

and from this follows

$$(\epsilon \otimes 1)\mathcal{F} = (\epsilon \otimes 1)e^f = 1 \otimes 1, \quad (107)$$

$$(1 \otimes \epsilon)\mathcal{F} = (1 \otimes \epsilon)e^f = 1 \otimes 1. \quad (108)$$

C. Cocycle condition

Cocycle condition is

$$(\mathcal{F} \otimes 1)(\Delta_0 \otimes 1)\mathcal{F} = (1 \otimes \mathcal{F})(1 \otimes \Delta_0)\mathcal{F} \quad (109)$$

We shall prove it using factorization properties of twist \mathcal{F}

$$(\Delta \otimes 1)\mathcal{F} = \mathcal{F}_{23}\mathcal{F}_{13} \quad (110)$$

$$(1 \otimes \Delta_0)\mathcal{F} = \mathcal{F}_{13}\mathcal{F}_{12} \quad (111)$$

where

$$\mathcal{F}_{12} = e^{\mathcal{K}_{\beta\alpha} \otimes L^{\alpha\beta} \otimes 1} \quad (112)$$

$$\mathcal{F}_{13} = e^{\mathcal{K}_{\beta\alpha} \otimes 1 \otimes L^{\alpha\beta}} \quad (113)$$

$$\mathcal{F}_{23} = e^{1 \otimes \mathcal{K}_{\beta\alpha} \otimes L^{\alpha\beta}}. \quad (114)$$

The first factorization property (110) can be proven to hold in a following way:

$$\begin{aligned} \mathcal{F}_{23}\mathcal{F}_{13} &= e^{1 \otimes (-ip_\alpha^W) \otimes l^\alpha} e^{-ip_\alpha^W \otimes 1 \otimes l^\alpha} \\ &= \left(e^{ip_\alpha^W \otimes 1 \otimes l^\alpha} e^{1 \otimes ip_\alpha^W \otimes l^\alpha} \right)^{-1} \\ &= e^{-i\mathcal{D}_\alpha^W(p^W \otimes 1, 1 \otimes p^W) \otimes l^\alpha} \end{aligned} \quad (115)$$

This holds because l_μ generates the same algebra as \hat{x}_μ and

$$e^{ik_1 \cdot \hat{x}} e^{ik_2 \cdot \hat{x}} = e^{i\mathcal{D}^W(k_1, k_2) \cdot \hat{x}}. \quad (116)$$

Furthermore, since $\mathcal{D}_\mu^W(p^W \otimes 1, 1 \otimes p^W) = \Delta p_\mu^W$, it follows that:

$$\mathcal{F}_{23}\mathcal{F}_{13} = e^{-i\Delta p_\alpha^W \otimes l^\alpha} = (\Delta \otimes 1)e^{-ip_\alpha^W \otimes l^\alpha} = (\Delta \otimes 1)\mathcal{F} \quad (117)$$

The second factorization property (111) for our twist follows trivially:

$$\begin{aligned} (1 \otimes \Delta_0)\mathcal{F} &= (1 \otimes \Delta_0)e^{\mathcal{K}^{\beta\alpha} \otimes L_{\alpha\beta}} \\ &= e^{\mathcal{K}^{\beta\alpha} \otimes (\Delta_0 L_{\alpha\beta})} \\ &= e^{\mathcal{K}^{\beta\alpha} \otimes 1 \otimes L_{\alpha\beta}} e^{\mathcal{K}^{\beta\alpha} \otimes L_{\alpha\beta} \otimes 1} \\ &= \mathcal{F}_{13}\mathcal{F}_{12} \end{aligned} \quad (118)$$

To see that cocycle condition follows from factorization properties, the first property, eq. (110), should be multiplied by \mathcal{F}_{12} from the right and second one, eq. (111), by \mathcal{F}_{23} from the left:

$$[(\Delta \otimes 1)\mathcal{F}](\mathcal{F} \otimes 1) = \mathcal{F}_{23}\mathcal{F}_{13}\mathcal{F}_{12} \quad (119)$$

$$(1 \otimes \mathcal{F})(1 \otimes \Delta_0)\mathcal{F} = \mathcal{F}_{23}\mathcal{F}_{13}\mathcal{F}_{12} \quad (120)$$

which implies

$$[(\Delta \otimes 1)\mathcal{F}](\mathcal{F} \otimes 1) = (1 \otimes \mathcal{F})(1 \otimes \Delta_0)\mathcal{F} \quad (121)$$

Since $[(\Delta \otimes 1)\mathcal{F}](\mathcal{F} \otimes 1) = (\mathcal{F} \otimes 1)(\Delta_0 \otimes 1)\mathcal{F}$, this is the cocycle condition (109).

D. R-matrix

R-matrix is defined by [34, 37]:

$$\mathcal{R} = \tilde{\mathcal{F}}\mathcal{F}^{-1} = e^{-il^\alpha \otimes p_\alpha^W} e^{ip_\beta^W \otimes l^\beta} = 1 \otimes 1 + r_{cl} + \mathcal{O}\left(\frac{1}{\kappa^2}\right) \quad (122)$$

where $\tilde{\mathcal{F}} = \tau_0 \mathcal{F} \tau_0$ is a transposed twist, see section VII. for details.

Up to the second order we have:

$$\ln \mathcal{R} = i(p_\alpha^W \otimes l^\alpha - l^\alpha \otimes p_\alpha^W) - \frac{1}{2} \left([p_\alpha^W, l^\beta] \otimes l^\alpha p_\beta^W - l^\alpha p_\beta^W \otimes [p_\alpha^W, l^\beta] \right) + \mathcal{O}(K^3) \quad (123)$$

where

$$[p_\mu^W, l_\nu] = [p_\mu^W, \hat{x}_\nu - x_\nu] = [p_\mu^W, \hat{x}^\alpha](\eta_{\alpha\nu} - \varphi_{\alpha\nu}^{-1}(p)) \quad (124)$$

Commutator $[p_\mu^W, \hat{x}_\nu]$ is given in equation (A21) and inverse matrices $\varphi_{\mu\nu}^{-1}$ are given in (19) and relation between p_μ and p_μ^W is given in equation (A20).

Generally, classical matrix $r_{cl} = \ln \mathcal{R}$ up to the first order in $\frac{1}{\kappa}$ and the classical r_{cl} matrices can be written in terms of $\mathfrak{gl}(n)$ generators as

$$r_{cl} = p_\mu \wedge l^\mu = -iK_{\beta\mu\alpha}p^\mu \wedge L^{\alpha\beta}, \quad (125)$$

where $K_{\beta\mu\alpha}$ are given in (17). Using (125) we find the classical r_{cl} -matrices for twists (96), (97), (98) and (99):

$$r_{cl}^{(\mathcal{C}_1)} = a \cdot p \wedge \left[\left(1 - \frac{c}{n}\right) D - c \frac{a_\alpha a_\beta}{a^2} S^{\alpha\beta} \right] \quad (126)$$

$$r_{cl}^{(\mathcal{C}_2)} = a \cdot p \wedge \left[\left(c - \frac{1}{n}\right) D - c \frac{a^\alpha a^\beta}{a^2} S_{\alpha\beta} \right] - (1-c)p^\alpha \wedge a^\beta \left(S_{\alpha\beta} + \frac{1}{2} M_{\alpha\beta} \right) \quad (127)$$

$$r_{cl}^{(\mathcal{C}_3)} = a \cdot p \wedge \left[\left(1 + \frac{1}{n}\right) D - c \frac{a_\alpha a_\beta}{a^2} S^{\alpha\beta} \right] + (1+c)p^\alpha \wedge a^\beta \left(S_{\alpha\beta} - \frac{1}{2} M_{\alpha\beta} \right) \quad (128)$$

$$r_{cl}^{(\mathcal{C}_4)} = a_\alpha P_\beta \wedge M^{\alpha\beta} \quad (129)$$

where

$$S_{\mu\nu} = \frac{1}{2}(L_{\mu\nu} + L_{\nu\mu}) - \frac{1}{n}D\eta_{\mu\nu} \quad (130)$$

is traceless symmetric part of $L_{\mu\nu}$, $D = L^\alpha_\alpha$ and $M_{\mu\nu} = L_{\mu\nu} - L_{\nu\mu}$. Note that for the case \mathcal{C}_4 , $r_{cl}^{(\mathcal{C}_4)}$ in (129) coincides with the r_{cl} for light-cone case discussed in [16]. Also $r_{cl}^{(\mathcal{C}_1)} = r_{cl}^{(\mathcal{C}_2)}$ for $c = 1$ and $r_{cl}^{(\mathcal{C}_1)} = r_{cl}^{(\mathcal{C}_3)}$ for $c = -1$, which is consistent with discussion in section III.

VI. TWISTED SYMMETRY ALGEBRAS

The family of twists (93), applied to undeformed $\mathfrak{gl}(n)$ Hopf algebra (subsection V A) produces the corresponding κ -deformed $\mathfrak{gl}(n)$ Hopf algebras. For $h \in \mathfrak{gl}(n)$, deformed coproduct Δh is related to undeformed coproduct $\Delta_0 h$ via:

$$\Delta h = \mathcal{F} \Delta_0 h \mathcal{F}^{-1} \quad (131)$$

In deformed $\mathfrak{gl}(n)$ Hopf algebra the coproduct Δ is:

$$\Delta p_\mu = \mathcal{F} \Delta_0 p_\mu \mathcal{F}^{-1} = p_\mu \otimes 1 + \Lambda_{\alpha\mu}^{-1} \otimes p^\alpha \quad (132)$$

$$\Delta L_{\mu\nu} = \mathcal{F} \Delta_0 L_{\mu\nu} \mathcal{F}^{-1} = L_{\mu\nu} \otimes 1 + \left(\Lambda_{\beta\gamma}^{-1} \frac{\partial \Lambda^\gamma_\alpha}{\partial p^\mu} p_\nu + \Lambda_{\beta\nu}^{-1} \Lambda_{\mu\alpha} \right) \otimes L^{\alpha\beta} \quad (133)$$

$$\Delta \Lambda_{\mu\nu} = \Lambda_{\mu\alpha} \otimes \Lambda^\alpha_\nu \quad (134)$$

$$\Delta(\Lambda^{-1})_{\mu\nu} = (\Lambda^{-1})_{\alpha\nu} \otimes (\Lambda^{-1})_\mu^\alpha \quad (135)$$

where $\Lambda_{\mu\nu}$ and $\Lambda_{\mu\nu}^{-1}$ are given in equations (71), (70), (76), (75), (81), (80), (86) and (85) for \mathcal{C}_1 , \mathcal{C}_2 , \mathcal{C}_3 and \mathcal{C}_4 respectively.

We point out that generators l_μ (see equation (7)) close κ -Minkowski algebra (see equation (10)) and $[l_\mu, p_\nu] = iK_{\alpha\mu\nu}p^\alpha$. Note that twists can be expressed in terms of l_μ and p_μ . From this it follows that $\Delta l_\mu = \mathcal{F}\Delta_0 l_\mu \mathcal{F}^{-1}$ (where $\Delta_0 l_\mu = l_\mu \otimes 1 + 1 \otimes l_\mu$) is closed in l_μ and p_μ .

The counit is unchanged:

$$\epsilon(p_\mu) = \epsilon(L_{\mu\nu}) = 0, \quad \epsilon(\Lambda_{\mu\nu}) = \epsilon(\Lambda_{\mu\nu}^{-1}) = \eta_{\mu\nu} \quad (136)$$

and the antipode S , obtained from coproduct and counit via $m[(S \otimes 1)\Delta h] = m[(1 \otimes S)\Delta h] = \epsilon(h)$, is given by

$$S(p_\mu) = -\Lambda_{\alpha\mu}^{-1}p^\alpha \quad (137)$$

$$S(L_{\mu\nu}) = -\left(\Lambda_{\beta}^{\gamma}\frac{\partial\Lambda_{\gamma\alpha}^{-1}}{\partial S(p^\mu)}S(p_\nu) + \Lambda_{\beta\nu}\Lambda_{\mu\alpha}^{-1}\right)L^{\alpha\beta} \quad (138)$$

$$S(\Lambda_{\mu\nu}) = \Lambda_{\mu\nu}^{-1} \quad (139)$$

$$S(\Lambda_{\mu\nu}^{-1}) = \Lambda_{\mu\nu} \quad (140)$$

The deformed Hopf algebra acting on $\hat{x}_\mu \otimes 1$, i.e. using $g\hat{f} = m\left[\Delta g(\blacktriangleright \otimes 1)(\hat{f} \otimes 1)\right]$, $\forall g \in \mathfrak{gl}(n)$ and $\hat{f} \in \hat{\mathcal{A}}$ leads to

$$[L_{\rho\sigma}, \hat{x}_\nu] = \eta_{\sigma\nu}\hat{x}_\rho + i\eta_{\sigma\nu}K_{\mu\rho\alpha}L^{\alpha\mu} - iK_{\mu\nu\sigma}L_\rho^\mu + iK_{\rho\nu\alpha}L_\sigma^\alpha \quad (141)$$

$$[p_\mu, \hat{x}_\nu] = -i(\eta_{\mu\nu} + K_{\beta\nu\mu}p^\beta) \quad (142)$$

which also leads to (2).

Let us consider special cases. For the case \mathcal{S}_1 , the twist operator is:

$$\mathcal{F}_{\mathcal{S}_1} = \exp\{-\ln(1 - a \cdot p) \otimes D\} \quad (143)$$

and coproducts and antipodes of p_μ , $D \equiv L_\alpha^\alpha$ and $M_{\mu\nu}$, obtained from the twist (143), are:

$$\begin{aligned} \Delta p_\mu &= p_\mu \otimes 1 + Z \otimes p_\mu = \Delta_0 p_\mu - a \cdot p \otimes p_\mu \\ \Delta D &= D \otimes 1 + Z^{-1} \otimes D \\ \Delta M_{\mu\nu} &= \Delta_0 M_{\mu\nu} + (a_\mu p_\nu - a_\nu p_\mu)Z^{-1} \otimes D \\ S(p_\mu) &= -Z^{-1}p_\mu \\ S(D) &= -ZD = -D + (a \cdot p)D \\ S(M_{\mu\nu}) &= -M_{\mu\nu} + (a_\mu p_\nu - a_\nu p_\mu) \end{aligned} \quad (144)$$

The coproduct and antipode of l_μ are:

$$\begin{aligned}\Delta l_\mu &= l_\mu \otimes 1 + Z^{-1} \otimes l_\mu \\ S(l_\mu) &= -Z l_\mu\end{aligned}\tag{145}$$

The symmetry of this case is Poincaré-Weyl symmetry. The case \mathcal{S}_1 corresponds to the right covariant realization $\hat{x}_\mu = x_\mu - a_\mu(x \cdot p)$, see equation (20), and is related to [38], but with interchanged left and right side in tensor product and with $a_\mu \rightarrow -a_\mu$.

For the case \mathcal{S}_2 , the twist operator is:

$$\mathcal{F}_{\mathcal{S}_2} = \exp \left\{ -a_\beta p_\alpha \frac{\ln(1 + a \cdot p)}{a \cdot p} \otimes L^{\alpha\beta} \right\}\tag{146}$$

and coproducts and antipodes of p_μ and $L_{\mu\nu}$ are:

$$\begin{aligned}\Delta p_\mu &= \Delta_0 p_\mu + p_\mu \otimes a \cdot p = p_\mu \otimes Z^{-1} + 1 \otimes p_\mu \\ \Delta L_{\mu\nu} &= \Delta_0 L_{\mu\nu} - a_\mu p^\alpha Z \otimes L_{\alpha\nu} \\ S(p_\mu) &= -Z p_\mu \\ S(L_{\mu\nu}) &= -L_{\mu\nu} - a_\mu p^\alpha L_{\alpha\nu}\end{aligned}\tag{147}$$

The coproduct and antipode of l_μ are:

$$\begin{aligned}\Delta l_\mu &= \Delta_0 l_\mu + a_\mu p_\alpha Z \otimes l^\alpha \\ S(l_\mu) &= -l_\mu - a_\mu(p \cdot l)\end{aligned}\tag{148}$$

The case \mathcal{S}_2 corresponds to the left covariant realization $\hat{x}_\mu = x_\mu [1 + (a \cdot p)]$, see equation (20).

For the case \mathcal{S}_3 , the twist operator is:

$$\mathcal{F}_{\mathcal{S}_3} = \exp \left\{ -\ln Z \otimes D + a_\alpha p_\beta^W \otimes L^{\alpha\beta} \right\}\tag{149}$$

where

$$p_\mu^W = \left(p_\mu + \frac{a_\mu p^2}{Z + 1 - a \cdot p} \right) \frac{\ln Z}{Z - 1}\tag{150}$$

and

$$Z = \sqrt{(1 - a \cdot p)^2 - a^2 p^2}\tag{151}$$

and coproduct and antipode of p_μ are:

$$\begin{aligned}\Delta p_\mu &= p_\mu \otimes 1 + \left[\eta_{\mu\alpha} + a_\mu \left(p_\alpha - \frac{a_\alpha p^2}{Z - 1 + a \cdot p} \right) \right] Z \otimes p^\alpha \\ S(p_\mu) &= - \left(p_\mu + a_\mu p^2 \frac{Z - 1}{Z - 1 + a \cdot p} \right) Z\end{aligned}\tag{152}$$

Similarly one finds $\Delta L_{\mu\nu}$ and $S(L_{\mu\nu})$ using equations (133) and (138) respectively. The coproduct and antipode of l_μ are:

$$\begin{aligned}\Delta l_\mu &= l_\mu \otimes 1 + Z \otimes l_\mu + \left[\frac{Z-1}{\ln Z} a_\mu p_\alpha^W + \frac{a_\alpha p_\mu}{Z^2} \right] \otimes l^\alpha \\ S(l_\mu) &= -Z^{-1} l_\mu + \frac{1-Z^{-1}}{\ln Z} a_\mu (p^W \cdot l) + \left(p_\mu + a_\mu p^2 \frac{Z-1}{Z-1+a \cdot p} \right) Z^3 (a \cdot l)\end{aligned}\tag{153}$$

The case \mathcal{S}_3 corresponds to $\hat{x}_\mu = x_\mu - a_\mu(x \cdot p) - (a \cdot x)p_\mu$, see equation (20).

For the case \mathcal{C}_4 , i.e. for the light-like κ deformation of Poincaré Hopf algebra, the twist operator is:

$$\mathcal{F}_{\mathcal{C}_4} = \exp \left\{ a_\alpha p_\beta \frac{\ln(1+a \cdot p)}{a \cdot p} \otimes M^{\alpha\beta} \right\}\tag{154}$$

and coproducts and antipodes of p_μ and $M_{\mu\nu}$, obtained from the twist (154), are:

$$\begin{aligned}\Delta p_\mu &= \Delta_0 p_\mu + \left[p_\mu a^\alpha - a_\mu \left(p^\alpha + \frac{1}{2} a^\alpha p^2 \right) Z \right] \otimes p_\alpha \\ \Delta M_{\mu\nu} &= \Delta_0 M_{\mu\nu} + (\delta_\mu^\alpha a_\nu - \delta_\nu^\alpha a_\mu) \left(p^\beta + \frac{1}{2} a^\beta p^2 \right) Z \otimes M_{\alpha\beta} \\ S(p_\mu) &= \left[-p_\mu - a_\mu \left(p_\alpha + \frac{1}{2} a_\alpha p^2 \right) p^\alpha \right] Z \\ S(M_{\mu\nu}) &= -M_{\mu\nu} + (-a_\mu \delta_\nu^\beta + a_\nu \delta_\mu^\beta) \left(p^\alpha + \frac{1}{2} a^\alpha p^2 \right) M_{\alpha\beta}\end{aligned}\tag{155}$$

The coproduct and antipode of l_μ are:

$$\begin{aligned}\Delta l_\mu &= \Delta_0 l_\mu + a_\mu p_\alpha \otimes l^\alpha \\ S(l_\mu) &= -l_\mu + a_\mu Z(p \cdot l)\end{aligned}\tag{156}$$

The case \mathcal{C}_4 corresponds to the natural realization $\hat{x}_\mu = x_\mu [1 + (a \cdot p)] - (a \cdot x)p_\mu$, see equation (18). It is the only solution compatible with κ -Poincaré Hopf algebra [16, 17, 29, 39].

VII. TRANSPOSED DRINFELD TWISTS AND LEFT-RIGHT DUAL κ -MINKOWSKI ALGEBRA

Transposed twist is $\tilde{\mathcal{F}} = \tau_0 \mathcal{F} \tau_0$, where $\tau_0 : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ is a linear map such that $\tau_0(A \otimes B) = B \otimes A \ \forall A, B \in \mathcal{H}$. It is obtained from \mathcal{F} by interchanging left and right side of tensor product, and it is also a Drinfeld twist satisfying normalization and cocycle condition. It is obtained from (91) by taking $t = 1$ and using transposed coproduct $\tilde{\Delta} p_\mu = \tau_0 \Delta p_\mu \tau_0$ instead of Δp_μ .

From transposed Drinfeld twist, a set of left-right dual generators of κ -Minkowski spacetime can be obtained:

$$\hat{y}_\mu = m \left[\tilde{\mathcal{F}}^{-1} (\triangleright \otimes 1) (x_\mu \otimes 1) \right] = x^\alpha \Lambda_{\mu\alpha}^{-1}\tag{157}$$

where $\Lambda_{\mu\nu}^{-1}$ for \mathcal{C}_1 , \mathcal{C}_2 , \mathcal{C}_3 and \mathcal{C}_4 is given in (70), (75), (80) and (85) respectively. For example, for cases \mathcal{S}_1 , \mathcal{S}_2 , \mathcal{S}_3 and \mathcal{C}_4 , generators \hat{y}_μ and \hat{x}_μ are:

$$\mathcal{S}_1: \hat{y}_\mu = x_\mu(1 - a \cdot p), \quad \hat{x}_\mu = x_\mu - a_\mu(x \cdot p), \quad (158)$$

$$\mathcal{S}_2: \hat{y}_\mu = x_\mu - (a \cdot x)p_\mu, \quad \hat{x}_\mu = x_\mu(1 + a \cdot p), \quad (159)$$

$$\mathcal{S}_3: \hat{y}_\mu = \left[x_\mu + a_\mu \left(x \cdot p - \frac{(a \cdot x)p^2}{Z + 1 - a \cdot p} \right) \right] Z, \quad \hat{x}_\mu = x_\mu - (a \cdot x)p - a_\mu(x \cdot p), \quad (160)$$

$$\mathcal{C}_4: \hat{y}_\mu = x_\mu + (a \cdot x)p_\mu - a_\mu \left(x \cdot p + \frac{a \cdot x}{2} p^2 \right) Z, \quad \hat{x}_\mu = x_\mu - (a \cdot x)p_\mu - x_\mu(a \cdot p), \quad a^2 = 0 \quad (161)$$

Generators \hat{y}_μ satisfy κ -Minkowski algebra but with $-a_\mu$ instead of a_μ [29]:

$$[\hat{y}_\mu, \hat{y}_\nu] = -i(a_\mu \hat{y}_\nu - a_\nu \hat{y}_\mu) \quad (162)$$

Generators of κ -Minkowski space \hat{x}_μ commute with their duals \hat{y}_μ [29]:

$$[\hat{x}_\mu, \hat{y}_\nu] = 0 \quad (163)$$

Generally, dual basis \hat{y}_μ is related to basis \hat{x}_μ via

$$\hat{y}_\mu = \hat{x}^\alpha (e^{-\mathcal{C}})_{\mu\alpha} \quad (164)$$

where $\mathcal{C}_{\mu\nu} = -C_{\mu\alpha\nu}(p^W)^\alpha$, where $C_{\mu\alpha\nu}$ are structure constants (see Appendix A in[36]). From this relation, and equations (5) and (157) for \hat{x}_μ and \hat{y}_μ respectively, it follows:

$$\Lambda_{\mu\nu}^{-1} = (e^{-\mathcal{K}})_{\mu\nu} = (e^{-\mathcal{C}})_{\mu\alpha} \varphi_\nu^\alpha \quad (165)$$

A. κ -Minkowski algebra from transposed twists with $a_\mu \rightarrow -a_\mu$

Starting with the family of twists (93), we define related Drinfeld twists $\tilde{\mathcal{F}}|_{a_\mu \rightarrow -a_\mu}$. They lead to nonlinear realizations of \hat{x}_μ , satisfying (2):

$$\hat{x}_\mu = m \left[\tilde{\mathcal{F}}^{-1} \Big|_{a_\mu \rightarrow -a_\mu} (\triangleright \otimes 1)(x_\mu \otimes 1) \right] = x^\alpha \Lambda_{\mu\alpha}^{-1} \Big|_{a_\mu \rightarrow -a_\mu} \quad (166)$$

Then the corresponding dual generators \hat{y}_μ are given by

$$\hat{y}_\mu = x_\mu + iK_{\beta\mu\alpha} L^{\alpha\beta} = x_\mu - l_\mu \quad (167)$$

Compared to case with transposed twists $\tilde{\mathcal{F}}$ in the beginning of this section, here the roles of \hat{x}_μ and \hat{y}_μ are interchanged, with $a_\mu \rightarrow -a_\mu$. With this new family of twists, \hat{x}_μ are non-linear realizations of κ -Minkowski space, while \hat{y}_μ are linear realizations of dual κ -Minkowski space.

If we apply twists $\tilde{\mathcal{F}}|_{a_\mu \rightarrow -a_\mu}$ to undeformed coproducts $\Delta_0 h$, we get coproducts $\tilde{\Delta}|_{a_\mu \rightarrow -a_\mu} h$, i.e. left and right side in coproducts Δh are interchanged and a_μ is replaced by $-a_\mu$.

We point out that solution \mathcal{C}_4 , and its transposed case, are of special interest because they lead to light-like κ -Poincaré Hopf algebra. They are related to the result by Borowiec and Pachoł [16] (comparison is given in section VIII B).

VIII. NON-LINEAR REALIZATIONS OF κ -MINKOWSKI SPACE AND RELATED DRINFELD TWISTS

We shall also present a few families of non-linear realizations and corresponding Drinfeld twist operators known in the literature so far.

A. Timelike deformations

The realizations we are considering are [24, 30]:

$$\hat{x}_i = x_i \varphi(A) \quad (168)$$

$$\hat{x}_0 = x_0 \psi(A) - a_0 x_k p^k \gamma(A) \quad (169)$$

where $A = -a \cdot p$ and functions $\varphi(A)$, $\psi(A)$ are such that $\varphi(0) = \psi(0) = 1$ and related to $\gamma(A)$ by:

$$\gamma(A) = \frac{\psi(A)}{\varphi(A)} \frac{d\varphi(A)}{dA} + 1 \quad (170)$$

Generically, the symmetry algebra is κ -deformed $\mathfrak{igl}(n)$ Hopf algebra. We will present two cases.

i) The first case is $\psi(A) = 1$, with arbitrary $\varphi(A)$ and $\gamma(A) = \frac{\varphi'(A)}{\varphi(A)} + 1$, see equation (170).

The coproducts of momenta are:

$$\Delta p_0 = \Delta_0 p_0 = p_0 \otimes 1 + 1 \otimes p_0 \quad (171)$$

$$\Delta p_i = \varphi(A \otimes 1 + 1 \otimes A) \left(\frac{p_i}{\varphi(A)} \otimes 1 + e^A \otimes \frac{p_i}{\varphi(A)} \right) \quad (172)$$

The twist operator is Abelian [35]

$$\mathcal{F}_\varphi = \exp \left\{ (N \otimes 1) \ln \frac{\varphi(A \otimes 1 + 1 \otimes A)}{\varphi(A \otimes 1)} + (1 \otimes N) \left(A \otimes 1 + \ln \frac{\varphi(A \otimes 1 + 1 \otimes A)}{\varphi(1 \otimes A)} \right) \right\} \quad (173)$$

where $N = ix_i p^i$ and $[N, A] = 0$. Since this twist is Abelian, it automatically satisfies cocycle condition, and therefore it is a Drinfeld twist. Special case is presented in [40, 41]

ii) In the second case, leading to Jordanian twists, given by Borowiec and Pachoł in [42], $\psi(A)$ is a linear function, i.e. $\psi(A) = 1 + rA$, where $r \in \mathbb{R}$, and $\gamma(A) = 0$, which leads to

$$\varphi = \psi^{-\frac{1}{r}} = (1 + rA)^{-\frac{1}{r}} \quad (174)$$

The coproducts of momenta are:

$$\Delta p_0 = p_0 \otimes \varphi(A) + 1 \otimes p_0 \quad (175)$$

$$\Delta p_i = p_i \otimes \psi(A) + 1 \otimes p_i \quad (176)$$

The family of corresponding twist operators is:

$$\mathcal{F}_r = \exp \left\{ \left(-L^0_0 + \frac{1}{r} L^k_k \right) \otimes \ln \varphi(A) \right\} \quad (177)$$

Special Jordanian twist was studied by Bu, Yee and Kim in [38] and corresponds to \mathcal{S}_1 , but with interchanged left and right side of the tensor product, and with $a_0 \rightarrow -a_0$ and $c = r + 1$, i.e.

$$\mathcal{F}_r = \tilde{\mathcal{F}}_{\mathcal{S}_1} \Big|_{a_0 \rightarrow -a_0, \ c=r+1} \quad (178)$$

B. Light-cone deformation

In the light-cone basis, the κ -Poincaré algebra was studied in [15] and [16], the corresponding twist is extended Jordanian twist, written in terms of two exponential factors. It is identical to transposed twist of $\mathcal{F}_{\mathcal{C}_4}$ with $a_\mu \rightarrow -a_\mu$, i.e. $\tilde{\mathcal{F}}_{\mathcal{C}_4} \Big|_{a_\mu \rightarrow -a_\mu}$.

Extended Jordanian twist corresponding to light-cone deformation is:

$$\mathcal{F}_{LC} = e^{-iM_{+-} \otimes \ln \Pi_+} e^{-\frac{i}{\kappa} M_{+a} \otimes P^a \Pi_+^{-1}} \quad (179)$$

where $[P_\mu, \hat{x}_\mu] = -i\eta_{\mu\nu} [1 + (a \cdot P)] + ia_\mu P_\nu$, see equation (18), and

$$\begin{aligned} \Pi_+ &= 1 + \frac{1}{\kappa} P_+ = 1 - a \cdot P \\ a_0 &= a_1 = \frac{1}{\sqrt{2}\kappa}, \quad a_j = 0 \text{ for } j > 1 \\ P_\pm &= \frac{P_0 \pm P_1}{\sqrt{2}} \\ M_{+-} &= iM_{01}, \quad M_{\pm j} = \frac{i}{\sqrt{2}} (M_{0j} \pm M_{1j}), \quad j > 1 \end{aligned} \quad (180)$$

If we define

$$\begin{aligned} \mathcal{A} &\equiv -iM_{+-} \otimes \ln \Pi_+ \\ \mathcal{B} &\equiv -\frac{i}{\kappa} M_{+a} \otimes P^a \Pi_+^{-1} \end{aligned} \quad (181)$$

then $[\mathcal{A}, \mathcal{B}] = \alpha \mathcal{B}$, where

$$\alpha \equiv 1 \otimes \ln \Pi_+ \quad (182)$$

and \mathcal{A} , \mathcal{B} and α generate algebra (C1), given in Appendix C. Using result (C2) from Appendix C, it follows that \mathcal{F}_{LC} , written as one exponential function, is given by

$$\mathcal{F}_{LC} = \exp \left\{ -i M_{+-} \otimes \frac{(\Pi_+ - 1) \ln \Pi_+}{\Pi_+ - 1} - \frac{i}{\kappa} M_{+a} \otimes P^a \frac{\ln \Pi_+}{\Pi_+ - 1} \right\} \quad (183)$$

Using our notation, this result is

$$\mathcal{F}_{LC} = \exp \left\{ M_{\alpha\beta} \otimes a^\alpha P^\beta \frac{\ln(1 - a \cdot P)}{a \cdot P} \right\} \quad (184)$$

which thus proves the relation

$$\mathcal{F}_{LC} = \tilde{\mathcal{F}}_{\mathcal{C}_4} \Big|_{a_\mu \rightarrow -a_\mu}. \quad (185)$$

The twist \mathcal{F}_{LC} leads to nonlinear realization (157) and the corresponding coproduct is transposed coproduct with $a_\mu \rightarrow -a_\mu$.

IX. OUTLOOK AND DISCUSSION

The full analysis of all possible linear realizations for κ -Minkowski space for time-, space- and light-like deformations is given. These realizations can be expressed in terms of the generators of $\mathfrak{gl}(n)$ algebra. Coproducts of momenta for linear realizations are constructed. We have presented a method for constructing Drinfeld twist operators corresponding to each linear realization of κ -Minkowski space and proved that it satisfies the cocycle and normalization conditions. We have constructed a whole new class of Drinfeld twists compatible with κ -Minkowski space and linear realizations, denoted by \mathcal{C}_1 , \mathcal{C}_2 , \mathcal{C}_3 and \mathcal{C}_4 . The symmetries generated by Drinfeld twists are described by κ -deformed $\mathfrak{igl}(n)$ -Hopf algebras, and in the special case of \mathcal{S}_1 and \mathcal{C}_4 we get the Poincaré-Weyl-Hopf algebra and light-like κ -Poincaré-Hopf algebra, respectively. We further illustrate how our method also works for constructing Drinfeld twists for nonlinear realizations and we compared our results to the examples already known in the literature.

In this paper we were dealing mostly with linear realizations and the corresponding Drinfeld twists. However, for any realization, in general, one can construct a twist operator that does not have to satisfy the cocycle condition in the Hopf algebra sense (i.e. not a Drinfeld twist), rather it satisfies the cocycle condition in a more general sense (up to tensor exchange identities [27]), i.e.

in the framework of Hopf algebroids [27]. It is crucial to notice that the κ -Minkowski space can be embedded into a Heisenberg algebra which has a natural Hopf algebroid structure. One can show that the star product resulting from this generalized twist operator is associative and that the corresponding symmetry algebra is a certain deformation of $\mathfrak{igl}(n)$ -Hopf algebra. This general framework is more suitable to address the questions of quantum gravity [43, 44] and related new effects to Planck scale physics.

The problem of finding all possible linear realizations is closely related to classification of bicovariant differential calculi on κ -Minkowski space [36]. Namely, the requirement that the differential calculus is bicovariant leads to finding all possible algebras between NC coordinates and NC one-forms that are closed (linear) in these NC one-forms. The corresponding equations for the structure constants (from the super-Jacobi identities) are exactly the same as eqs. (8,9). The linear realizations elaborated in this paper are expressed in terms of Heisenberg algebra, but one can extend this to super-Heisenberg algebra, by introducing Grassmann coordinates and momenta. This way one can construct the extended twists [36, 45] which have the same desired properties, but also give the whole differential calculi.

With linear realization it is much easier to understand and to perform practical calculation in the NC space. In [46] it is proposed that the NC metric should be a central element of the whole differential algebra (generated by NC coordinates and NC one-forms). This NC metric should encode some of the main properties of the quantum theory of gravity. We hope that using the tool of linear realizations one can perform such calculations for a large class of deformations, and for all types of bicovariant differential calculi and predict new contributions to the physics of quantum black holes and the quantum origin of the cosmological constant [47].

Recently [18], the Drinfeld twist corresponding to \mathcal{C}_4 was analyzed and the corresponding scalar field theory was discussed. We are planing to further analyse the properties of quantum field theories [48], especially gauge theories that arise from this twist, but we are also interested in pursuing further investigations on the physical aspects of $\mathcal{C}_{1,2,3}$ cases.

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Appendix A: Derivation of coproduct Δp_μ

Here we present construction of equations for $K_\mu(k)$ and $P_\mu(k_1, k_2)$ and their solutions for linear realizations.

From equation (46) we find

$$e^{-i\lambda k_1 \cdot \hat{x}} p_\mu e^{i\lambda k_1 \cdot \hat{x}} \triangleright e^{ik_2 \cdot x} = P_\mu(\lambda k_1, k_2) e^{ik_2 \cdot x} \quad (\text{A1})$$

where $(k_1)_\mu, (k_2)_\mu \in \mathbb{M}^n$ and $p_\mu \in \mathcal{T}$. Differentiating both sides by $\frac{\partial}{\partial \lambda}$ and using $\hat{x}_\mu = x^\alpha \varphi_{\alpha\mu}(p)$ we get the relation between $P_\mu(\lambda k_1, k_2)$ and realization $\varphi_{\mu\nu}(P(\lambda k_1, k_2))$:

$$\frac{\partial P_\mu(\lambda k_1, k_2)}{\partial \lambda} = \varphi_{\mu\alpha}(P(\lambda k_1, k_2)) k_1^\alpha \quad (\text{A2})$$

Note that for $\lambda = 0$ the boundary condition is

$$P_\mu(0, k) = k_\mu. \quad (\text{A3})$$

The coproduct for momentum p_μ is calculated by [31–33]:

$$\Delta p_\mu = \mathcal{D}_\mu(p \otimes 1, 1 \otimes p) \quad (\text{A4})$$

where the function $\mathcal{D}_\mu(k_1, k_2)$ is given by

$$\mathcal{D}_\mu(k_1, k_2) = P_\mu(K^{-1}(k_1), k_2) \quad (\text{A5})$$

and $K_\mu^{-1}(k_1)$ is inverse function of $P_\mu(k_1, 0) = K_\mu(k_1)$.

The function $\varphi_{\mu\alpha}(p)$ describes the choice of realization in a following way:

$$\hat{x}_\mu = x^\alpha \varphi_{\alpha\mu}(p) \quad (\text{A6})$$

In the case of linear realizations $\varphi_{\alpha\mu}(p)$ is:

$$\varphi_{\alpha\mu} = \eta_{\alpha\mu} + K_{\beta\mu\alpha} p^\beta \quad (\text{A7})$$

therefore

$$\frac{\partial P_\mu(\lambda k_1, k_2)}{\partial \lambda} = (k_1)_\mu + K^\alpha_{\beta\mu} P_\alpha(\lambda k_1, k_2) k_1^\beta \quad (\text{A8})$$

This can be solved by expanding $P(\lambda k_1, k_2)$ in terms of λ .

$$P_\mu(\lambda k_1, k_2) = \sum_{n=0}^{\infty} P_\mu^{(n)}(k_1, k_2) \lambda^n, \quad (\text{A9})$$

which, comparing the terms with the same power of λ , leads to

$$P_\mu^{(1)}(k_1, k_2) = (k_1)_\mu + K^\alpha{}_{\beta\mu} P_\alpha^{(0)}(k_1, k_2) k_1^\beta, \quad (\text{A10})$$

$$P_\mu^{(n+1)}(k_1, k_2) = \frac{1}{n+1} K^\alpha{}_{\beta\mu} P_\alpha^{(n)}(k_1, k_2) k_1^\beta, \text{ for } n \geq 1. \quad (\text{A11})$$

Boundary condition (A3) leads to:

$$P_\alpha^{(0)}(k_1, k_2) = (k_2)_\mu. \quad (\text{A12})$$

For sake of brevity, let us define:

$$\mathcal{K}_{\mu\nu}(k) \equiv -K_{\mu\alpha\nu} k^\alpha. \quad (\text{A13})$$

Using (A12) and (A13), equations (A10) and (A11) become:

$$P_\mu^{(1)}(k_1, k_2) = (k_1)_\mu - \mathcal{K}_{\alpha\mu}(k_1) k_2^\alpha, \quad (\text{A14})$$

$$P_\mu^{(n+1)}(k_1, k_2) = -\frac{1}{n+1} \mathcal{K}^\alpha{}_\mu(k_1) P_\alpha^{(n)}(k_1, k_2), \text{ for } n \geq 1. \quad (\text{A15})$$

The solution for $P_\mu(k_1, k_2)$ is:

$$P_\mu(k_1, k_2) = \left(\frac{\eta - e^{-\mathcal{K}(k_1)}}{\mathcal{K}(k_1)} \right)_{\alpha\mu} k_1^\alpha + \left(e^{-\mathcal{K}(k_1)} \right)_{\alpha\mu} k_2^\alpha \quad (\text{A16})$$

Solution for $K_\mu(k) = P_\mu(k, 0)$ is simply:

$$K_\mu(k) = \left(\frac{\eta - e^{-\mathcal{K}(k)}}{\mathcal{K}(k)} \right)_{\alpha\mu} k^\alpha \quad (\text{A17})$$

It is useful to define

$$k_\mu^W = K_\mu^{-1}(k) \quad (\text{A18})$$

Inserting this definition and solution (A16) into (A5) we get

$$\mathcal{D}_\mu(k_1, k_2) = P_\mu(k_1^W, k_2) = (k_1)_\mu + \left(e^{-\mathcal{K}(k_1^W)} \right)_{\alpha\mu} k_2^\alpha. \quad (\text{A19})$$

The momentum $p_\mu^W = K_\mu^{-1}(p)$, introduced in section IV C (see eq. (66)), is related to p_μ via

$$p_\mu = \left(\frac{\eta - e^{-\mathcal{K}}}{\mathcal{K}} \right)_{\alpha\mu} (p^W)^\alpha \quad (\text{A20})$$

and is given in closed form in (72), (77), (82) and (87) for the solutions \mathcal{C}_1 , \mathcal{C}_2 , \mathcal{C}_3 and \mathcal{C}_4 respectively.

The momentum p_μ^W corresponds to Weyl symmetric ordering [23, 24, 30]

$$[p_\mu^W, \hat{x}_\nu] = \eta_{\mu\nu} \frac{a \cdot p^W}{e^{-a \cdot p^W} - 1} + \frac{a_\nu p_\mu^W}{a \cdot p^W} \left(1 + \frac{a \cdot p^W}{e^{-a \cdot p^W} - 1} \right). \quad (\text{A21})$$

For p_μ^W , it is useful to define:

$$\mathcal{K}_{\mu\nu} \equiv \mathcal{K}_{\mu\nu}(p^W) = -K_{\mu\alpha\nu}(p^W)^\alpha. \quad (\text{A22})$$

Using this definition, and solution (A19) with the equation (A4), finally leads to the coproduct:

$$\Delta p_\mu = p_\mu \otimes 1 + \left(e^{-\mathcal{K}} \right)_{\alpha\mu} \otimes p^\alpha \quad (\text{A23})$$

Appendix B: Construction of the twist operator from the coproduct of momenta

It can be shown that for any $A_{\mu\nu}$ such that $[A_{\mu\nu}, L_{\sigma\rho}] = 0$ and $[A_{\mu\nu}, A_{\sigma\rho}] = 0$, the following holds:

$$: e^{L_\beta^\alpha A_\alpha^\beta} := e^{L_\beta^\alpha [\ln(1+A)]_\alpha^\beta} \quad (\text{B1})$$

This identity is a generalization of the result presented in [30] in equations (A. 16) and (A. 17). See also Section 2 in [49].

Twists can be calculated from the known coproducts of momenta using the equation (91). We would like to write the twist in the following form:

$$\mathcal{F} = e^f, \quad (\text{B2})$$

where

$$f = \sum_{s=1}^{\infty} f_s, \quad (\text{B3})$$

and $f_s \in \mathcal{U}[\mathfrak{gl}(n)] \otimes \mathcal{U}[\mathfrak{gl}(n)]$ is contribution to f in s -th order of $\frac{1}{\kappa}$.

Inserting (55) into (91), for $t = 0$ we get:

$$\mathcal{F}^{-1} =: \exp \left\{ (\Lambda^{-1} - 1)^\beta_\alpha \otimes L^\alpha_\beta \right\} : \quad (\text{B4})$$

From equation (B1) it follows

$$\mathcal{F}^{-1} = \exp \left\{ (\ln \Lambda^{-1})^\beta_\alpha \otimes L^\alpha_\beta \right\} \quad (\text{B5})$$

Since $\Lambda_{\mu\nu} = (e^{\mathcal{K}})_{\mu\nu}$, we find the twist:

$$\mathcal{F} = \exp \left(\mathcal{K}_{\beta\alpha} \otimes L^{\alpha\beta} \right) \quad (\text{B6})$$

Since $\mathcal{K}_{\mu\nu} = -K_{\mu\alpha\nu}(p^W)^\alpha$ and $l_\mu = -iK_{\beta\mu\alpha}L^{\alpha\beta}$, the result can also be written as:

$$\mathcal{F} = \exp \left(-ip_\alpha^W \otimes l^\alpha \right) \quad (\text{B7})$$

Appendix C: A special case of the BCH formula

Let us consider algebra generated by \mathcal{A} , \mathcal{B} and α :

$$[\mathcal{A}, \mathcal{B}] = \alpha\mathcal{B}, \quad [\mathcal{A}, \alpha] = [\mathcal{B}, \alpha] = 0 \quad (\text{C1})$$

then

$$e^{\mathcal{A}}e^{\mathcal{B}} = e^{\mathcal{A}+\mathcal{B}f(\alpha)} \quad (\text{C2})$$

where

$$f(\alpha) = \frac{\alpha}{1 - e^{-\alpha}} \quad (\text{C3})$$

This can be proved by representing \mathcal{A} and \mathcal{B} with 2 by 2 matrices:

$$\mathcal{A} = \frac{\alpha}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (\text{C4})$$

These matrices satisfy the algebra (C1). Their exponentials are:

$$e^{\mathcal{A}} = \begin{pmatrix} e^{\frac{\alpha}{2}} & 0 \\ 0 & e^{-\frac{\alpha}{2}} \end{pmatrix}, \quad e^{\mathcal{B}} = 1 + \mathcal{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (\text{C5})$$

leading to

$$e^{\mathcal{A}}e^{\mathcal{B}} = \begin{pmatrix} e^{\frac{\alpha}{2}} & e^{\frac{\alpha}{2}} \\ 0 & e^{-\frac{\alpha}{2}} \end{pmatrix}. \quad (\text{C6})$$

On the other hand, since

$$\mathcal{A} + \mathcal{B}f(\alpha) = \begin{pmatrix} \frac{\alpha}{2} & f(\alpha) \\ 0 & -\frac{\alpha}{2} \end{pmatrix} \quad (\text{C7})$$

it follows that

$$e^{\mathcal{A}+\mathcal{B}f(\alpha)} = \begin{pmatrix} e^{\frac{\alpha}{2}} & \frac{f(\alpha)}{\alpha}(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}) \\ 0 & e^{-\frac{\alpha}{2}} \end{pmatrix} \quad (\text{C8})$$

Comparing (C6) and (C8) gives $f(\alpha)$ in (C3).

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